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Bounds of Laplacian spectrum of graphs based on the domination number

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Abstract

Let G be a connected graph of order n. A dominating set in G is a subset S of $V(G)$ such that each element of $V(G) - S$ is adjacent to a vertex of S. The least cardinality of a dominating set is the domination number. In the paper, we will give bounds of the Laplacian spectrum of G involving the domination number.

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1. Introduction

Let $G = (V, E)$ be a simple undirected graph with n vertices. For $v \in V$, we use $N(v)$ to denote the neighbors of v, that is, the set of all vertices in G adjacent to v. For

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a subgraph H of G, let $N_H(v) = N(v) \cap V(H)$ and $|N_H(v)| = d_H(v)$. If $H = G$, then $N_H(v)$ and $d_H(v)$ are written as $N(v)$ and $d(v)$ respectively. Let $\Delta(G)$ and $\delta(G)$ be the maximum and minimum degree of vertices of G, respectively. Let $S \subseteq V(G)$. Denote by $G[S]$ the subgraph of G induced by S. In the case of no confusion, we write $N_S(v)$ and $d_S(v)$ instead of $N_{G[S]}(v)$ and $d_{G[S]}(v)$, respectively. A dominating set in G is a subset X of $V(G)$ such that each element of $V(G) - X$ is adjacent to at least one vertex of X. The least cardinality of a dominating set is the domination number of G, denoted by $\gamma(G)$. A set Z of vertices of a graph G is called a cover of G if every edge of G is incident to at least one vertex of Z. The least cardinality of a cover of G is called the covering number of G and denoted by $\tau(G)$. It is easy to see that a cover of G is also a dominating set of G. Thus we get

$$\gamma(G) \leqslant \tau(G).$$

Let $A(G)$ be the adjacency matrix of G and $D(G) = \mathrm{diag}(d(v_1), d(v_2), \ldots, d(v_n))$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. Clearly, $L(G)$ is a real symmetric matrix. From this fact and Geršgorin's Theorem, it follows that its eigenvalues are nonnegative real numbers. The eigenvalues of an $n \times n$ matrix M are denoted by $\lambda_1(M), \lambda_2(M), \ldots, \lambda_n(M)$, while for a graph G, we will use $\lambda_i(G) = \lambda_i$ to denote $\lambda_i(L(G))$, $i = 1, 2, \ldots, n$ and assume that $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_{n-1}(G) \geqslant \lambda_n(G)$. It is well known that $\lambda_n(G) = 0$ and the algebraic multiplicity of zero as an eigenvalue of $L(G)$ is exactly the number of connected components of G [11]. In particular, the second smallest eigenvalue $\lambda_{n-1}(G) > 0$ if and only if G is connected. This leds Fiedler [2] to define it as the algebraic connectivity of G. The eigenvalues, $\lambda_1(G)$ (also call the Laplacian spectral radius of G) and $\lambda_{n-1}(G)$, have received a great deal of attention (see, for example [3,7–8,10–13,15]). In this paper, we give bounds of the Laplacian spectral radius and algebraic connectivity of G involving the domination number $\gamma(G)$.

2. Lemmas

Lemma 1 [5,6]. *Let M be a real symmetric matrix with eigenvalues $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n$. Given a partition $\{1, 2, \ldots, n\} = \triangle_1 \cup \triangle_2 \cup \cdots \cup \triangle_m$ with $|\triangle_i| = n_i > 0$, consider the corresponding blocking $M = (M_{ij})$, so that M_{ij} is an $n_i \times n_j$ block. Let e_{ij} be the sum of the entries in M_{ij} and put $B = (e_{ij}/n_i)$ (i.e., e_{ij}/n_i is an average row sum in M_{ij}). Let $\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_m$ be the eigenvalues of B. Then the inequalities*

$$\mu_i \geqslant \gamma_i \geqslant \mu_{n-m+i} \qquad (i = 1, 2, \ldots, m)$$

hold. Moreover, if for some integer k, $1 \leqslant k \leqslant m$, $\mu_i = \gamma_i$ for $i = 1, 2, \ldots, k$ and $\mu_{n-m+i} = \gamma_i$ for $i = k+1, k+2, \ldots, m$, then all the blocks M_{ij} of M have constant row and column sums.

Let G be a graph and X a dominating set of G. For any $v \in X$, denote

$$W(v, X) = \{u \in V(G) - X | N(u) \cap X = \{v\}\}.$$

We first show the following useful lemma.

Lemma 2. *Let G be a connected graph with domination number $\gamma(G)$. Then there exists a dominating set X of G with $|X| = \gamma(G)$ such that for each $v \in X$, $|W(v, X)| \geqslant 1$.*

Proof. Let X be a dominating set of G with $|X| = \gamma(G)$. Choose X such that $\sum_{v \in X} d_X(v)$ is as large as possible. Next, we will show that $|W(v, X)| \geqslant 1$.

Suppose there exists a vertex $v \in X$ such that $W(v, X) = \emptyset$, i.e., $N_X(u) - \{v\} \neq \emptyset$ for all $u \in V(G) - X$. Thus we have $N(v) \cap X = \emptyset$, (otherwise, $X' = X - \{v\}$ is a dominating set of G, a contradiction). Set $N(v) = \{w_1, w_2, \ldots, w_l\}$. Then $l \geqslant 1$ by G being connected. By assumption, $N_X(w_i) - \{v\} \neq \emptyset$ for $i = 1, 2, \ldots, l$. Let $X' = (X \cup \{w_1\}) - \{v\}$. Then X' is a dominating set of G with $|X'| = \gamma(G)$ but $\sum_{v \in X'} d_{X'}(v) > \sum_{v \in X} d_X(v)$ (since $d_{X'}(w_1) > d_X(v) = 0$), a contradiction with the choice of X. $\square$

Lemma 3. *Let $G = (V, E)$ be a connected graph of order n and G_1 be an induced subgraph of G with n_1 ($n_1 < n$) vertices and average degree r_1(i.e., $r_1 = \sum_{v \in V(G_1)} d_{G_1}(v)/n_1$). Set $d_1 = \sum_{v \in V(G_1)} d(v)/n_1$. Then*

$$\lambda_1(G) \geqslant \frac{n(d_1 - r_1)}{n - n_1}.$$

Moreover, if the equality holds, then $d_{G_2}(u) = s$ for all vertex $u \in V(G_1)$ and $d_{G_1}(v) = t$ for all vertex $v \in V(G_2)$, where $G_2 = G[V - V(G_1)]$.

Proof. Set $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(G_2) = \{v_{n_1+1}, v_{n_1+2}, \ldots, v_n\}$. Let $r_2 = \sum_{v \in V(G_2)} d_{G_2}(v)/(n - n_1)$ and $d_2 = \sum_{v \in V(G_2)} d(v)/(n - n_1)$. Rewrite $L(G)$ as

$$L(G) = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} D_{11} - A(G_1) & -A_{12} \\ -A_{21} & D_{22} - A(G_2) \end{pmatrix},$$

where $D_{11} = \mathrm{diag}(d(v_1), d(v_2), \ldots, d(v_{n_1}))$ and $D_{22} = \mathrm{diag}(d(v_{n_1+1}), d(v_{n_1+2}), \ldots, d(v_n))$. For $1 \leqslant i, j \leqslant 2$, let e_{ij} be the sum of the entries in L_{ij} and put $B = (e_{ij}/n_i)$. Then

$$B = \begin{pmatrix} d_1 - r_1 & r_1 - d_1 \\ r_2 - d_2 & d_2 - r_2 \end{pmatrix},$$

and then $|\lambda I - B| = \lambda(\lambda - d_1 - d_2 + r_1 + r_2)$. Thus by Lemma 1,

$$\lambda_1(G) \geqslant d_1 + d_2 - r_1 - r_2. \tag{1}$$

Note that $n_1(d_1 - r_1) = (n - n_1)(d_2 - r_2)$, and hence

$$d_1 + d_2 - r_1 - r_2 = n(d_1 - r_1)/(n - n_1).$$

By (1), we have

$$\lambda_1(G) \geqslant \frac{n(d_1 - r_1)}{n - n_1}.$$

If the equality holds, then $\lambda_1(G) = \lambda_1(B)$. Since $\lambda_n(G) = \lambda_2(B) = 0$, we have A_{12} and A_{21} have constant row and column sums by Lemma 1. This implies $d_{G_2}(u) = s$ for all vertex $u \in V(G_1)$ and $d_{G_1}(v) = t$ for all vertex $v \in V(G_2)$. $\square$

Let $G = (V, E)$ be a graph and X a nonempty subset of V. The edge density of X is given by

$$\rho(X) = \frac{|V||E_X|}{|X||X^c|},$$

where $X^c = V - X$ and E_X is the set of all edges with one end in X and the other end in X^c. In [1], the following result was given.

Lemma 4 [1]. *Let $G = (V, E)$ be a graph. For any nontrivial subset X of V, the edge density of X satisfies*

$$\lambda_{n-1}(G) \leqslant \rho(X) = \frac{|V||E_X|}{|X||X^c|}.$$

Moreover, if a graph G satisfies the equality for some cut E_X, then there are integers s and t such that the following condition must hold:

(A) *Each vertex in X is adjacent to s vertices in X^c and every vertex in X^c is adjacent to t vertices in X, and*

(B) $s|X| = t|X^c|$.

The following three lemmas will be used in the proof of our main theorems.

Lemma 5 [4]. *Let G be a graph with at least one edge. Then*

$$\lambda_1(G) \geqslant \Delta(G) + 1.$$

Moreover, if G is connected, then the equality holds if and only if $\Delta(G) = |V(G)| - 1$.

Lemma 6 [2]. *Let G be a graph of order n. Then*

$$\lambda_{n-1}(G) \leqslant \delta(G).$$

Lemma 7 [9]. *Let G be a connected graph of order n. If $\delta(G) \geqslant 2$ and $G \notin \{B_i | 1 \leqslant i \leqslant 7\}$, then $\gamma(G) \leqslant \frac{2}{5}n$, where B_i $(1 \leqslant i \leqslant 7)$ are illustrated in Fig. 1.*

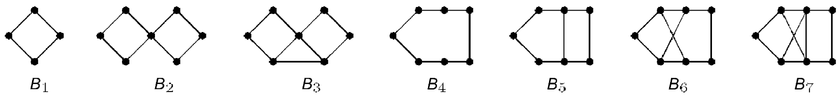

Fig. 1.

3. Main results

Now, we will present our main results.

Theorem 8. *Let G be a connected graph of order $n \geqslant 2$. Then*

$$\lambda_{n-1}(G) \leqslant \frac{n(n-2\gamma(G)+1)}{n-\gamma(G)}.$$

Moreover, the equality holds if and only if G is isomorphic to the complete bipartite graph $K_{2,2}$.

Proof. By Lemma 2, let X be the dominating set of G with $|X| = \gamma(G)$ such that for each $v \in X$, $|W(v, X)| \geqslant 1$. Thus we have

$$\begin{aligned}|E_X| &\leqslant \sum_{v\in X} |W(v,X)| + |X|\left(n - |X| - \sum_{v\in X}|W(v,X)|\right)\\ &= \gamma(G)(n-\gamma(G)) - (\gamma(G)-1)\sum_{v\in X}|W(v,X)|\\ &\leqslant \gamma(G)(n-\gamma(G)) - \gamma(G)(\gamma(G)-1),\end{aligned}$$

and hence

$$|E_X| \leqslant (n-2\gamma(G)+1)\gamma(G). \tag{2}$$

Thus, by Lemma 4, we have

$$\lambda_{n-1}(G) \leqslant \frac{n(n-2\gamma(G)+1)}{n-\gamma(G)}. \tag{3}$$

Suppose that the equality of (3) holds. Then we first have that $\lambda_{n-1}(G) = \rho(X)$. So G satisfies the conditions (A) and (B) of Lemma 4. Now by (2) we have that $|W(v, X)| = 1$ for all vertex $v \in X$. Thus by (A) of Lemma 4, we have $t = 1$ and $V = X \cup (\cup_{v\in X} W(v, X))$, and hence $n = 2\gamma(G)$ and $s = 1$ by (B) of Lemma 4. From $n = 2\gamma(G)$ and (3), we get $\lambda_{n-1}(G) = 2$. By Lemma 6, $\delta(G) \geqslant 2$. Thus we have that $G \in \{B_i | 1 \leqslant i \leqslant 7\}$ by Lemma 7, where B_i is the graph listed in Fig. 1. Noting that $\gamma(B_1) = \frac{n}{2}$ and for $i = 2, 3, \ldots, 7$, $\gamma(B_i) = \frac{3}{7}n < \frac{n}{2}$, we get G is isomorphic to $B_1 = K_{2,2}$.

It is easy to see that if G is isomorphic to $K_{2,2}$, then $\lambda_3(K_{2,2}) = 2$ and $\gamma(K_{2,2}) = 2$. Thus, the equality of Theorem 8 holds obviously. $\square$

Since $\lambda_{n-1}(G) > 0$ if G is connected, we have $\gamma(G) < \frac{n+1}{2}$ by Theorem 8. Since $\gamma(G)$ is an integer number, we get the following corollary by (3).

Corollary 9 [14]. *Let G be a connected graph of order $n \geqslant 2$. Then*

$$\gamma(G) \leqslant \frac{n}{2}.$$

Theorem 10. *Let $G = (V, E)$ be a connected graph of order $n \geqslant 2$. Then*

$$\lambda_1(G) \geqslant \frac{n}{\gamma(G)}. \tag{4}$$

Equality holds if and only if $\Delta(G) = n - 1$.

Proof. Let X be the dominating set of G with $|X| = \gamma(G)$. Denote $X^c = V - X$. By the definition of the dominating set, we have that

$$|E_X| \geqslant n - \gamma(G). \tag{5}$$

By Lemma 3, we have

$$\lambda_1(G) \geqslant \frac{n(d_1 - r_1)}{n - \gamma(G)} \tag{6}$$

where $d_1 = \frac{1}{\gamma(G)} \sum_{v \in X} d(v)$ and $r_1 = \frac{1}{\gamma(G)} \sum_{v \in X} d_X(v)$. By (5), we have that

$$\begin{aligned} d_1 - r_1 &= \frac{1}{\gamma(G)} \left(\sum_{v \in X} d(v) - \sum_{v \in X} d_X(v) \right) \\ &= \frac{1}{\gamma(G)} |E_X| \geqslant \frac{1}{\gamma(G)} (n - \gamma(G)). \end{aligned} \tag{7}$$

By (6), we have $\lambda_1(G) \geqslant \frac{n}{\gamma(G)}$.

In order for the equality to hold, all inequalities in the above argument should be equalities. By Lemma 3, we have $d_{X^c}(u) = s$ for all vertex $u \in X$ and $d_X(v) = t$ for all vertex $v \in X^c$. By (5), we have $t = 1$.

If $\Delta(G) < n - 1$, then $\lambda_1 > \Delta(G) + 1$ by Lemma 5. Since $\gamma(G) = \frac{n}{\lambda_1(G)}$, we have

$$\gamma(G) < \frac{n}{\Delta(G) + 1}. \tag{8}$$

Note that

$$\gamma(G) = |X|, \quad n = |X| + |X^c| \quad \text{and} \quad s|X| = |X^c|.$$

Thus, by (8), we have $|X| < (s + 1)|X|/(\Delta(G) + 1)$, and hence $s = d_{X^c}(u) > \Delta(G)$, a contradiction. This implies that if the equality holds, then $\Delta(G) = n - 1$.

Conversely, let G be a graph of order n with $\Delta(G) = n - 1$, then $\gamma(G) = 1$, and hence the equality of (4) holds by Lemma 5. □

Corollary 11. *Let G be a connected graph of order $n \geqslant 2$. Then*

(i) [16] $\lambda_1(G) \geqslant \frac{n}{\tau(G)}$.
(ii) *Equality holds if and only if G is a star, i.e. G is isomorphic to $K_{1,n-1}$.*

Proof. Since $\gamma(G) \leqslant \tau(G)$, (i) follows from Theorem 10.

Suppose that the equality of (i) holds, by Theorem 10, we have $\Delta(G) = n - 1$ and $\gamma(G) = \tau(G) = 1$. Therefore G is a star. Conversely, if G is a star, then $\tau(G) = 1$ and $\lambda_1(G) = n$, and hence (ii) holds. □

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